Studies on surfaces of general type with irregularity one

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学位の種類	博　士（理　学）
学位記番号	理博第2151号
学位授与年月日	平成17年3月25日
学位授与の要件	学位規則第4条第1項該当
研究科，専攻	東北大学大学院理学研究科（博士課程）数学専攻
学位論文題目	Studies on surfaces of general type with irregularity one （不正則数1の一般型曲面の研究）
論文審査委員	（主査）教授 石田正典 教授 雪江明彦，尾形庄悦

論文内容要旨

Let S be a minimal surface of general type. The most basic invariants are the *genus* and the *irregularity*, defined by

$$p_g(S) = \dim H^2(S, \mathcal{O}_S) \quad \text{and} \quad q(S) = \dim H^1(S, \mathcal{O}_S).$$

Let K_S be the canonical divisor of S. We have an additional invariant K_S^2 which denotes the self-intersection number of K_S.

In this paper, we mainly consider the case where $q(S) = 1$. We assume that S is a minimal surface of general type with $q(S) = 1$. Let $\alpha : S \longrightarrow \mathrm{Alb}(S)$ be the Albanese map of S. By the assumption $q(S) = 1$, we see that $\mathrm{Alb}(S)$ is an elliptic curve. Hence, surface S with $q(S) = 1$ naturally has a fibration over an elliptic curve $E = \mathrm{Alb}(S)$.

Horikawa [Ho3], Catanese and Ciliberto [Ca1, C-C1, C-C2] studied surfaces of general type with $p_g(S) = q(S) = 1$ and $K_S^2 = 2, 3$. Let g be the genus of fibers of the Albanese map α. When $K_S^2 = 2$, Horikawa [Ho3], Catanese and Ciliberto [Ca1, C-C1] showed that $g = 2$. When $K_S^2 = 3$, Catanese and Ciliberto [C-C1, C-C2] showed that $g = 2$ or 3. Moreover, in the case where $g = 3$, they showed that α gives a non-hyperelliptic fibration and such surfaces are given by relative quartic hypersurfaces of a certain $\mathbb{P}^2$-bundle.

For a fibration $f : S \longrightarrow C$, we call a fiber containing critical points of f a *singular fiber*. In Chapter 1, by considering the structure of surfaces with $p_g(S) = q(S) = 1$, $K_S^2 = 3$, $g = 3$, we study singular fibers of the Albanese maps of these surfaces. By a numerical argument, the number of singular fibers of α is at least one and less than or equal to nine. It seems that, in general, this fibration has nine singular fibers. Conversely, surfaces with $p_g(S) = q(S) = 1$, $K_S^2 = 3$, $g = 3$, which have a unique singular fiber are most special. We consider mainly such surfaces and show the following theorem.

Theorem 1 *For an arbitrary elliptic curve E, there exists a surface S which has no (-2)-curves and satisfies $p_g(S) = q(S) = 1$, $K_S^2 = 3$, $g = 3$, $\mathrm{Alb}(S) \cong E$. If E has an automorphism of complex multiplication type, then there exist exactly two isomorphism classes of such surfaces. Otherwise, they have exactly four isomorphism classes.*

We know by observing monodromies that a fibration over $\mathbb{P}^1$ which has only one singular fiber does not exist.

On the other hand, there may exist a surface over an elliptic curve with a unique singular fiber. Really, the surfaces in Theorem 1 shows that existence of such fibrations.

Xiao [X1] studied fibrations of genus two and obtained the inequality $2 \leq K_S^2 \leq 6$ for a minimal surface S of general type with $p_g = q = 1$, $g = 2$. He also showed the existence of a surface S with $K_S^2 = 4$. It is well known the existence of a minimal surface S of general type with $p_g = q = 1$, $g = 2$ and $K_S^2 = 2, 3$ in [Ca1, C-C1]. Catanese [Ca2] showed the existence of a surface of general type with $p_g = q = 1$, $g = 2$ and $K_S^2 = 5$.

In Chapter 2, we study the minimal surfaces of general type with $p_g = q = 1$, $g = 2$ and $K_S^2 = 4, 5$. In particular, we concentrate on the study of the structure of the Albanese fibration over the elliptic curve Alb(S). We classify surfaces with $p_g = q = 1$, $g = 2$ and $K_S^2 = 4$ (resp. 5) into three types IV_1, IV_2 and IV_3 (resp. V_1, V_2 and V_3). In this classification, surfaces constructed by Xiao and Catanese are surfaces of type IV_1 and type V_3, respectively. Therefore, one can ask whether surfaces of type IV_2, IV_3 and V_2 exist or not. In Chapter 2, we will show the following:

Theorem 2 *For any elliptic curve E, there exist surfaces S of types* IV_2, IV_3 *and* V_2 *satisfying the property* Alb*(S)* $\cong$ *E.*

In addition, by using our method, we also construct surfaces of type V_3 which are different from these of Catanese.

Let S be an algebraic surface of general type over $\mathbb{C}$ with a relatively minimal fibration f over a nonsingular curve C of genus g. Similarly as in surfaces, invariants of fibrations are defined. Let K_S and K_C be the canonical divisors of S and C, respectively. Set $K_{S/C} = K_S \otimes (f^* K_C)^{\vee}$. Set $\chi_f = \deg f_* K_{S/C}$ and $K_f^2 = K_{S/C}^2$. Xiao [X3] studied algebraic surfaces with a fibration. He defined $\lambda(f) = K_f^2/\chi_f$ and calls it the *slope* of f. He showed also the slope inequality $\frac{4(g-1)}{g} \leq \lambda(f)$ if f is not locally trivial. When $\lambda(f) < 4$, by using this inequality, we see the upper bound $4/(4-\lambda(f))$ of g. If f is non-hyperelliptic fibration and if the relative canonical bundle is semi-stable, then it follows from Konno's slope inequality [Kn] that $\frac{5(g-6)}{g} \leq \lambda(f)$. When $\lambda(f) < 5$, by using this inequality, we see the upper bound $6/(5-\lambda(f))$ of g. On the other hand, we do not know the bounds for the genus g of fibers of f for a hyperelliptic fibration f with $4 \leq \lambda(f)$. In Chapter 3, we consider the case where $\lambda(f) = 4$ and prove the following:

Theorem 3 *Let g be an integer greater than three. We set*

$$\chi(g) = \begin{cases} \dfrac{g}{2} - 1 & \text{if } g \text{ is even,} \\ g-3 & \text{if } g \text{ is odd and greater than four,} \\ 1 & \text{if } g = 3. \end{cases}$$

Then any relatively minimal hyperelliptic fibrations $f : S \longrightarrow C$ of genus g with $\lambda(f) = 4$ satisfy $\chi_f \geq \chi(g)$. Moreover, there exists a relatively minimal hyperelliptic fibration f of genus g with $\lambda(f) = 4$ satisfying $\chi_f = \chi(g)$.

For any integer g greater than three, we will show that there exist minimal fibrations f of genus g such that χ_f is lowest value $\chi(g)$ of our estimate. Especially, fibrations constructed in the proof of Theorem 3 satisfy the condition

$q=1$.

Let S be a minimal surface of general type with $q(S) = 1$. We assume that the Albanese map α of S gives the hyperelliptic fibration with slope four. By applying this theorem to S, we obtain the following:

Corollary 4 *Let S be a minimal surface of general type with $q = 1$, $g \neq 2$ and $K_S^2 = 4p_g(S)$ whose Albanese map α gives the hyperelliptic fibration. Then :*

(i) If $p_g(S) = 1$ then we have $g = 3, 4$.

(ii) If $p_g(S) = 2$ then we have $g = 3, 4, 5, 6$.

Moreover, there exist surfaces in each case.

In the proof of Theorem 3, we see the existence of such surfaces. Note that Xiao [X1] showed the existence of surfaces with $p_g(S) = z$ and $g = 2$ for $z = 1, 2$.

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論文審査の結果の要旨

不正則数が1である一般型代数曲面について，不変量の取り得る値が限定され，実際にその不変量をもつ曲面が存在するかどうかが問題となる。本研究は，アルバネーゼ写像の構造を詳しく調べることにより，単に不変量をもつ代数曲面の存在を示すだけでなく，同じ不変量をもつ代数曲面の構造を決定し，構成することで存在を証明することを目的としている。

一般型曲面は，エンリケスと小平による代数曲面の分類において，その構造を統一的には扱えないものとして分類された。そこで，曲面の不変量をいくつか選び，それらの取り得る値の範囲を細分して，その部分に不変量をもつ代数曲面特有の構造を調べるという研究が続いている。特に，代表的な堀川の研究では，ある種の曲線束の構造を標準的にもつ曲面が調べられた。本研究の対象は，不正則数が1の一般型代数曲面であるから，アルバネーゼ写像が楕円曲線上の曲線束の構造を標準的に与えている。

本研究の結果は，大きく3部に分かれる。第1部では，楕円曲線を任意に与えたとき，アルバネーゼ写像がその曲線上の種数3の曲線束の構造を与える代数曲面を構成した。特に，特異ファイバーをひとつだけ持つものの同型類をすべて求めた。

第2部では，ファイバーの種数が2で標準因子の自己交点数が4または5の曲面の構造を詳しく調べて分類し，新しい曲面の例を構成した。以前に知られていた例がこの分類のどこに属するかも調べた。

第3部では，アルバネーゼ写像のファイバーが超楕円曲線であるという仮定の下で，ファイバーの種数の取り得る上限を求めた。さらに，種数が上限と一致する曲面の存在も示した。

曲線束の構造を持つ代数曲面の研究では，不正則数が0である場合に多くの研究がなされ，不正則数が1の研究は数編のみであったが，本研究は，この方面の研究を大きく前進させた。本研究結果は，自立して研究活動を行うに必要な高度の研究能力と学識を有することを示している。したがって，石田弘隆提出の博士論文は，博士（理学）の学位論文として合格と認める。